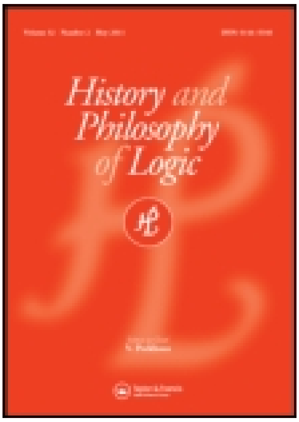

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Frege, Dedekind, and the Origins of Logicism

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Frege, Dedekind, and the Origins of Logicism

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This paper has a two-fold objective: to provide a balanced, multi-faceted account of the origins of logicism; to rehabilitate Richard Dedekind as a main logicist. Logicism should be seen as more deeply rooted in the development of modern mathematics than typically assumed, and this becomes evident by reconsidering Dedekind's writings in relation to Frege's. Especially in its Dedekindian and Fregean versions, logicism constitutes the culmination of the rise of 'pure mathematics' in the nineteenth century; and this rise brought with it an inter-weaving of methodological and epistemological considerations. The latter aspect illustrates how philosophical concerns can grow out of mathematical practice, as opposed to being imposed on it from outside. It also sheds new light on the legacy and the lasting significance of logicism today.

Logicism is the thesis that all of mathematics, or core parts of it, can be reduced to logic. This is an initial, rough characterization, since it leaves open, among others, what 'logic' is meant to encompass and how it is to be characterized. A main goal of the present paper is to broaden our horizon in both respects. A second goal is to rehabilitate Richard Dedekind as an important logicist. In historical surveys of logicism, he is only mentioned occasionally; instead, Gottlob Frege and Bertrand Russell typically take center stage, i.e. they are seen as the two original and primary logicists.[1] There has also been a renaissance of logicism recently, based on works by Crispin Wright, Bob Hale, and others. For these 'neo-logicists', the main reference point is again Frege, since they see themselves as reviving a Fregean position. By juxtaposing Frege's and Dedekind's approaches, I will argue that a corresponding Dedekindian position deserves to be reconsidered as well. More generally, the paper is meant to illustrate that adopting a historical perspective can help in clarifying both the origins and the philosophical significance of logicism.[2]

When approaching logicism historically today, one does not have to start from scratch. A number of relevant studies have appeared recently, concerning the development of mathematics, philosophy, and their relation in the nineteenth century. I will draw heavily on these studies, while adding a few historical and philosophical observations of my own.[3] My main contribution will consist, then, in showing how several earlier, only partly connected insights from the literature can be integrated into a more balanced, multi-faceted whole. The resulting synthesis is meant to support the following related theses: Logicism should be seen as more deeply rooted in the development of modern mathematics than is often assumed; this becomes evident if we reconsider Dedekind's writings, including their relationship

[1] Recent exceptions are *Stein 1998*, *Demopoulos and Clark 2007*; cf. also *Ferreirós, forthcoming*. Much earlier, Dedekind was taken to be a main logicist in *Cassirer 1907, 1910* as well.

[2] My goals overlap with those of *Kitcher 1986*. However, since its publication much has been learned that I will try to incorporate. Thus, I will disagree with Kitcher on various basic points.

[3] I am especially indebted to work on Frege by Mark Wilson and Jamie Tappenden, and to work on Dedekind by Howard Stein, José Ferreirós, and Ernst Cassirer (cf. later footnotes). However, I will explore commonalities between Frege and Dedekind more than they do; and I will try to establish closer ties to current debates in the philosophy of mathematics, e.g. concerning neo-logicism.

to Frege's; while Dedekind's and Frege's approaches differ in important respects, there are also significant commonalities, especially for present purposes; in particular, both in its Dedekindian and Fregean versions logicism constitutes the culmination of the rise of 'pure mathematics' in the nineteenth century; this rise brought with it an inter-weaving of methodological and epistemological concerns; and by analyzing the latter, one can see how philosophical concerns can grow out of mathematical practice, as opposed to being imposed on it from outside.

The paper is structured as follows: First some standard assumptions about logicism and its history will be made explicit (Section 1). This will be supplemented, as further background, by a summary of Dedekind's relevant contributions (Section 2). Then we turn to the issue of whether Dedekind should be seen as a logicist, indeed a main representative of logicism, starting with testimony by his contemporaries (Section 3). After that, we will compare two suggestions for what the general source of logicism was: certain debates in nineteenth-century philosophy (Section 4) or parallel developments in nineteenth-century mathematics (Section 5). In further probing the latter, two significant aspects will be the rise of a more 'conceptual' methodology for mathematics (Section 6) and the emerging use of classes during the period (Section 7). After having thus painted a richer, more comprehensive picture of the origins of logicism, we will be in a position to re-evaluate Dedekind's contributions, as well as the prospects for a Dedekindian form of neo-logicism (Section 8). A few concluding remarks will round things off (Section 9).

1. Standard assumptions about logicism and its history

Let me begin with what I take to be the received view about the origins and the fate of logicism. This view has been challenged before, including in the literature on which I will build; but it remains influential. It will constitute the foil for my own account. I will not argue that it is completely wrong, but that it is incomplete and misleading in certain respects, thus in need of supplementation and adjustment.

The thesis that mathematics is reducible to logic is usually taken to combine two sub-claims: first, there is the claim that all mathematical concepts are definable in terms of logical concepts; and second, there is the claim that all mathematical truths are derivable from logical truths. Understood as such, logicism is frequently seen as a contribution to certain inner-philosophical debates. As Kant had argued famously, both mathematical concepts and mathematical truths depend on 'intuition', not just on logic; and consequently, mathematics is 'synthetic *a priori*'. While Kant's position was not universally accepted, it cast a huge shadow – almost every subsequent philosopher felt compelled to respond to it, in one way or another. Among Kant's opponents in the nineteenth century, J.S. Mill stood out. For Mill, mathematics, like all our knowledge, is empirical. When seen against that background, the significance of logicism is two-fold: It aims to establish, against Kant, that intuition is not essential to mathematics; but it also offers a defense, against Mill and other empiricists, of the *a priori* character of mathematics. In fact, both Frege and Russell motivated logicism explicitly along such lines.[4]

While Russell became the main proponent of logicism in the twentieth century, he granted historical priority to Frege. What inaugurated logicism as a serious option, for both, was the introduction of modern relational and quantificational logic, in the form of a theory of types, as first presented in *Frege's Begriffsschrift* (*1879*). In the later parts of *Begriffsschrift*, Frege applied this new logic explicitly to establish that certain kinds of arithmetic reasoning presumed to depend on intuition can be treated purely logically (mathematical induction,

[4] For a philosophically subtle discussion of Frege's project as seen from this angle, cf. *Weiner 2004*. For a very recent systematic treatment of Russell along the same general lines, cf. *Korhonen 2013*.

especially). In *Grundlagen der Arithmetik* (*1884*), he expanded on the motivation of his logicist project in more philosophical terms. While refuting Kant's, Mill's, and related views was not his only goal, it was the primary one (or so the story goes). In particular, Frege now used a variant of Kant's analytic-synthetic distinction for framing his discussion. The basic goal was, consequently, to show that all arithmetic truths are 'analytic'. Finally, in his *magnum opus*, *Grundgesetze der Arithmetik* (*1893/1903*), Frege added a theory of classes ('extensions of concepts') to his logical system, as this seemed to provide the best, and perhaps the only, way to achieve his logicist objective.

Frege's theory of classes in *Grundgesetze* is inconsistent, of course, as Russell informed him in a famous letter from 1902. Nevertheless, Russell shared Frege's logicist goal; he too wanted to show that mathematics (now all of it) is 'part of logic'. In his first work on the topic, *Principles of Mathematics* (*1903*), Russell thus hailed Frege's work as trailblazing, even if it was inadequate in its details. He also suggested a way of circumventing the antinomy and similar problems: the use of a more intricate ('ramified') theory of logical types than the one Frege had used. Russell's background and motivation were not identical with Frege's; among others, he was much more centrally motivated by questions about certainty than Frege. Yet he too emphasized the opposition to Kant and Mill, or more generally, to Kant-inspired idealism and to descendants of British empiricism.[5] Then again, Russell did not draw the conclusion that the reduction of mathematics to logic established the former to be 'analytic'. Instead, he thought that it showed both logic and mathematics to be 'synthetic', in his sense of those terms.[6]

In *Principia Mathematica* (1910–1913), his own *magnum opus* (co-written with A.N. Whitehead), Russell developed his logicism further, both technically and philosophically. Parallel to Frege's mature system, *Principia* contains not just a form of relational and quantificational logic, but also a theory of classes. Or at least, it contains a way of simulating the presence of classes, while Russell worked with a 'no-classes theory of classes' in the end (where reference to classes is explained away in terms of his theory of descriptions). As is well known, also incorporated were several new axioms about whose 'logical' status Russell was unsure himself: a version of the axiom of infinity, the axiom of choice (the 'multiplicative axiom'), and most controversially, the axiom of reducibility. These axioms had to be added so as to make possible, within Russell's more restrictive type theory, a revised logicist reduction of arithmetic, analysis, and other parts of mathematics.

To be able to acknowledge directly that *Principia Mathematica* establishes a version of logicism, these three additional axioms had to be acceptable as 'logical'. But this was quickly called into question (by Wittgenstein and others). Russell's way out was to treat the controversial axioms not as outright truths, but as the antecedents of conditionals, so that mathematical theorems turn into logically provable if-then statements. But this led to further questions. Still, Russellian logicism continued to have defenders. Most prominently, Rudolf Carnap – assuming that a satisfactory defense of Russell's system as 'logical' could still be found – represented it at the famous Königsberg conference in 1930. Carnap's presentation starts as follows:

> Logicism is the thesis that mathematics is reducible to logic, hence nothing but a part of logic. Frege was the first to espouse this view (*1884*). In their great work, *Principia Mathematica*, the English mathematicians A.N. Whitehead and B. Russell

[5] Cf. *Hylton 1990* for a reading of Russell that emphasizes his reaction to British idealism. The central role of certainty for Russell is analyzed in *Proops 2006*. For a recent account of the origins of Russell's logicism that ties it more to mathematical developments, cf. *Gandon 2012*.

[6] For a discussion of Frege's and Russell's divergent notions of 'analytic' and 'synthetic', as well as other differences, cf. *Kremer 2006*. I will come back to this issue in connection with Frege later.

produced a systematization of logic from which they constructed mathematics. (*Carnap 1931*, p. 91)

In this passage, we have both a brief characterization of logicism and an influential statement of its standard history – with Frege as logicism's pioneer and *Principia Mathematica* as its high point. For Carnap, unlike for Russell, the goal was again to establish the 'analyticity' of mathematics, albeit in yet another sense of that term. Carnap saw logicism also again as primarily a response to Kant, or better, to related views in nineteenth- and early twentieth-century German Neo-Kantianism.[7]

By the 1930–1940s, Frege's original version of logicism was seen as refuted by Russell's antinomy, while Russell's variant faced unresolved questions about his additional axioms. Carnap inherited the latter problem. In addition, his attempts to explicate the notion of analyticity came to be seen as problematic as well (after Quine's criticisms, among others). From the 1950s to the 1980s, logicism was thus widely seen as a failure. This changed with Crispin Wright's *Frege's Conception of Numbers as Objects* (*1983*), in which a Fregean form of logicism was brought back into play. Since then, Wright's 'neo-logicism' has been elaborated in great detail, also by other 'neo-Fregeans'. The crucial twist now is to start with 'Hume's Principle' and similar 'abstraction principles', as opposed to basing everything on a theory of classes. Yet this is still meant to be Fregean in several other respects, including: the background logic is Frege's simple theory of types; and the derivation of arithmetic from Hume's Principle is recovered from his writings. In addition, the central (but strongly contested) claim is that this shows arithmetic to be reducible to logic plus some 'quasi-definitional' principles, and in that sense, to be 'analytic' (although the latter term is often avoided because of Quine's legacy).

As Fregean neo-logicism has been discussed in great detail elsewhere, I will leave it at this rough sketch.[8] My main goal in the present paper is not to argue for or against it; similarly for Frege's, Russell's, and Carnap's earlier versions. Crucial for me is, instead, that the stereotypical story about logicism that we just rehearsed – the one entrenched by Russell and Carnap, and still accepted by both neo-Fregeans and many of their critics today – unjustly ignores another relevant figure: Richard Dedekind. Moreover, this story presents logicism primarily, and often exclusively, as a response to Kant, Mill, and other philosophers.[9] It thus focuses our attention on a particular inner-philosophical debate, often tied to 'analyticity' or related notions (and their fate). In contrast, I now want to place logicism into a richer context, both philosophically and mathematically. Reconsidering Dedekind is important, among others, because it leads naturally in that direction. As a result, both Frege and Fregean neo-logicism will appear in a new light as well.

2. Dedekind's foundational works: a brief summary and some guiding suggestion

Dedekind's works were not entirely ignored by his contemporaries, including Frege and Russell. Nor were their successors in philosophy – from Carnap through W.V.O. Quine and Michael Dummett to today – not aware of his technical achievements. It is also hard to deny that some of these philosophers articulated seemingly strong reasons for doubting that he

7 Carnap's adoption of logicism, as part of logical empiricism, constituted partly a reaction against his own initial neo-Kantian leanings; cf. *Richardson 1998* and *Friedman 2000*. Another important thinker who emphasized the anti-Kantian thrust of logicism, highlighted Russell's role in it, and, thus, contributed strongly to the standard view about logicism was Louis Couturat; cf. *Couturat 1905*.

8 For further discussions of neo-logicism that focus on its philosophical significance, cf. *Demopoulos 1995*, *Boolos 1999*, *Hale and Wright 2001*, *MacBride 2003*, and *Heck 2012*.

9 With respect to Frege, this aspect has been challenged before, e.g. in *Benacerraf 1981*; cf. also the response in *Weiner 1984*. My own account will differ significantly from both.

should be seen as a main representative of logicism. But before weighing in against these reasons, let me provide a brief reminder of Dedekind's most relevant contributions.[10] My summary of them, together with my discussion of methodological concerns that guide his approach overall, will lead to the following suggestion (to be substantiated further in later sections): Logicism in general, and Dedekind's version in particular, is much more deeply rooted in mathematical developments than the standard story suggests.

Dedekind's main foundational writings – the ones on which the case for seeing him as a logicist can draw primarily – are two small booklets: *Stetigkeit und irrationale Zahlen 1872* and *Was sind und was sollen die Zahlen?* (*1888*).[11] As suggested by its title, in his 1872 essay, Dedekind discusses irrational numbers, such as $\sqrt{2}$, together with the central notion of continuity (or line-completeness); or rather, he considers the real numbers in general. In his 1888 essay, Dedekind studies the nature and role of the natural numbers, and thus, the deeper foundations of 'the science of numbers' in a broad sense. In both cases, he provides crucial re-conceptualizations. Not only for the standards of his time, Dedekind's approach is very abstract. From today's point of view, it looks set-theoretic and model-theoretic, partly even category-theoretic; it also involves a form of mathematical structuralism.[12]

A good way to understand the goal of *Dedekind's 1872* essay is that it provides a systematic treatment of both rational and irrational numbers, in such a way that the results of the Differential and Integral Calculus, or of mathematical 'analysis', can be derived rigorously from basic concepts. No such treatment had been available before. How did people proceed earlier, then? Dedekind puts it as follows:

> [T]he way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitude – which is nowhere carefully defined – and explains number as the result of measuring such a magnitude by another of the same kind. (*Dedekind 1963*, pp. 9–10)

An initial problem with the earlier approach is, as noted here, that the crucial notion of magnitude is never defined carefully. But a deeper problem lurks behind that, as Dedekind goes on to argue. Namely, the notion of ratio between two magnitudes 'can be clearly developed only after the introduction of irrational numbers' (p. 10, fn.*). In other words, the notion of irrational number is foundationally prior – it has to be in place for giving a clear, general treatment of ratios, not *vice versa*.[13] A third problem is that the old approach breaks down when we go from the real to the complex numbers, i.e. complex numbers cannot be thought of in terms of magnitudes. Yet they play an important role in 'the theory of numbers' too, as various extensions of the Calculus have shown.

Beyond what was said so far, how did mathematicians reason about magnitudes, including about the limit processes central to the Calculus, if not in terms of carefully defined concepts? They had 'recourse to geometric evidences', as Dedekind notes (*1963*, p. 1). Now, he is not totally opposed to the use of such evidence; as he admits: '[E]ven now such recourse to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint' (*Dedekind 1963*). But for systematic and truly 'scientific' purposes, such recourse is problematic. This is partly for the reasons

[10] For further details concerning Dedekind's foundational works, cf. *Reck 2003*, also *Sieg and Schlimm 2005*. For a more general perspective on him, cf. *Reck 2008/2011*, as well as *Ferreirós 1999*.

[11] In *Reck 2008/2011*, I argue that it is misleading to sharply separate Dedekind's foundational from his other contributions. Later sections of the present essay will reconfirm this point.

[12] For the ways in which Dedekind's approach is set-, model-, and category-theoretic, cf. *Ferreirós 1999* and *Corry 2004*. For a philosophical discussion of his structuralism, cf. *Reck 2003*.

[13] For the relation between Dedekind's approach to the real numbers and the earlier (Eudoxean) theory of ratios, cf. *Stein 1990*. I will come back to benefits of Dedekind's approach below.

already given, but also because an appeal to geometry introduces 'foreign notions' into arithmetic and analysis. And why is the latter problematic? Because it makes it impossible to discover what the 'vital points' in the corresponding proofs are (p. 3), and thus, what the 'true origin' of the theorems at issue are (p. 2). What we need, instead, is 'to bring out clearly the corresponding purely arithmetic properties' (p. 5).

I have dwelt on several methodological remarks in *Stetigkeit und irrationale Zahlen* for the following reason: Clearly, it is crucial for Dedekind to be able to deal with the real numbers without appeal to 'intuition'. However, his corresponding arguments do not appeal to Kant, Mill, or inner-philosophical debates. Instead, they involve issues of mathematical explicitness, precision, and progress (filling in gaps, avoiding circularities, methodological 'purity', etc.).[14] Having rejected a geometrically based approach, Dedekind proceeds as follows instead: His starting point is the system of rational numbers, conceived of as an ordered field. By considering cuts on that field ('Dedekind cuts'), he can define the notion of continuity 'purely arithmetically'. It is also not hard to show, on that basis, that the system of rational numbers, while dense, is not continuous (a point left obscure before). Dedekind then constructs the system of all cuts, as a new mathematical object. Relative to an induced field structure and ordering, this system can be shown to be continuous. Finally, 'the real numbers' are introduced by 'abstraction', as the ordered field isomorphic to the system of cuts whose elements have only structural properties.

From a later perspective, it is natural to reconstruct Dedekind's procedure, as just described, in set-theoretic terms (except for the last step, involving 'Dedekind abstraction'). But in 1872, he has not yet reflected systematically on his relevant techniques and on the basic notions underlying them. That is one aspect added in his 1888 essay, *Was sind und was sollen die Zahlen?* Here Dedekind starts from three very general, abstract notions: those of object (*Ding*), set (*System*), and function (*Abbildung*). Crucially, he sees all three as part of 'logic'. Early in the Preface of the essay he announces, correspondingly, that he will develop 'that part of logic which deals with the theory of numbers' (*Dedekind 1963*, p. 31). He adds:

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notions of intuition of space and time, that I consider it an immediate result from the laws of thought. (*Dedekind 1963*)

Here Dedekind's goal of distancing arithmetic from geometric intuition and basing it, instead, on 'the laws of thought' is formulated explicitly. But once again, Kant, Mill, etc. are not mentioned. The logicist project is justified methodologically, and with clear reference back to his work on the real numbers, as follows:

> It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number domain that we are prepared to accurately investigate our notions of space and time [. . .]. (*Dedekind 1963*)

The basic argument is this: We need arithmetic and, ultimately, 'logical' notions to ground geometric notions (e.g. by distinguishing denseness and continuity), not *vice versa*. Dedekind supports this point further by noting that all of Euclidean geometry holds in a space where we restrict ourselves to points corresponding to algebraic numbers, thus a space that is not continuous (*Dedekind 1963*, pp. 37–38.)

[14] For a helpful further discussion of this side of Dedekind's methodology, cf. *Detlefsen 2011*. See also the corresponding remarks about Dedekind in *Tappenden 2005, 2006, 2008*.

As already mentioned, the fundamental 'logical' notions in *Dedekind's 1888* essay are those of object, set, and function. (Presumably 'abstraction' is also part of logic; more on it below.) While all three are crucial, the notion of function is singled out further in connection with the foundations or arithmetic. As Dedekind writes:

> If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. (*Dedekind 1964*, p. 32)

As it is so tempting to think of Dedekind's contributions in set-theoretic terms, it is worth emphasizing in this context that he does not reduce functions to sets. Based on passages like this, one might even wonder whether it is the other way around for him, i.e. whether he thinks about sets in terms of functions in the end, somehow. But Dedekind does not clarify this issue fully. Also, officially all three of his 'logical' notions are fundamental. (Again, more on this point later.)

Utilizing those three notions, Dedekind re-conceptualizes the natural numbers as follows in his 1888 essay: He starts by defining the notions of infinity (being 'Dedekind-infinite') and simple infinity (being a model of the Dedekind-Peano axioms, essentially). After an infamous 'proof' that there are infinite sets (Proposition 66), thus also simple infinities, he shows that the latter are all isomorphic, which implies that the same theorems hold for them. Next, and starting with an already constructed simple infinity, Dedekind introduces 'the natural numbers' by 'abstraction', along structuralist lines, parallel to how he introduced the real numbers in his earlier essay. He also analyzes recursive definitions and proofs by mathematical induction 'logically' (via his theory of 'chains'). This allows him, finally, to define addition and multiplication, as well as to establish how his sequence of natural numbers, introduced as ordinal numbers, can be used to measure the cardinality of finite sets (by using initial segments as tallies).

3. Dedekind's logicism: toward a historical reevaluation

I noted initially that Dedekind is only sometimes mentioned in histories of logicism today while Frege and Russell take precedence, as Carnap's influential discussion illustrates. This was not always the case. To begin with, while Dedekind does not use the term 'logicism' anywhere (nor does Frege), we already saw that in his own eyes his project was one of showing that arithmetic is 'a part of logic'. Moreover, when he became aware of Frege's parallel project – by reading Frege's *Grundlagen der Arithmetik*, between the first (1888) and second (1893) edition of his own essay on the natural numbers – his reaction was: '[Frege] stands upon the same ground with me' (*Dedekind 1963*, p. 43). That is to say, Dedekind saw his and Frege's projects as closely related, or as being in an important sense of the same kind, even though he was well aware of differences with respect to details.

Beyond Dedekind, there were others who characterized his project in logicist terms. In fact, in the late nineteenth century it appears to have been Dedekind, not Frege, who was seen as the main representative of logicism. Remarks by three major figures may suffice as evidence. Ernst Schröder, the main German member of the Boolean school in logic, was one of the first to carefully study *Was sind und was sollen die Zahlen?*, as his *Vorlesungen über die Algebra der Logik* (*1890/1895*) show. He also talks about the temptation to join 'those who, like Dedekind, consider arithmetic a branch of logic'. In lectures on geometry from 1899, David Hilbert remarks: 'As a given we take the laws of pure logic and in particular of arithmetic. (On the relationship between logic and arithmetic see Dedekind, *Was sind und was sollen die Zahlen?*)' Finally, C.S. Peirce, another pioneer of logic, notes in 1911 that Dedekind's theory of sets and functions constitutes 'an early and significant acknowledgment that the so-called "logic of relatives" is an integral part of logic'.[15]

As another authority in this context we can appeal to – Frege. While Dedekind became aware of Frege's *Grundlagen* only after the publication of his *1888* essay, as he tells us, there is evidence that Frege read *Was sind und was sollen die Zahlen?* right after its publication. In fact, he taught a seminar on it in *1889/1890*, probably as an occasion to absorb its content more fully. This was highly unusual for him.[16] Frege started commenting on Dedekind's essay in print in *Grundgesetze, Vol. 1*. In the latter's Preface, he remarks: '(*Dedekind 1888*) is the most thorough work on the foundations of arithmetic that has come to my attention in recent years' (*Frege 1893*, p. 196). Coming from someone who usually finds little to praise in other people's writings, this is a striking statement. Frege goes on: 'Dedekind too is of the opinion that the theory of numbers is a part of logic' (*Frege 1893*, p. 196). The view that Frege and Dedekind 'stand on the same ground' in this connection was thus mutual.

Frege would not be Frege if he did not also criticize Dedekind, in both volumes of *Grundgesetze*. I take his two most apt criticisms in Volume 1 to be the following. First, the conciseness of Dedekind's proofs does not always allow one to be sure that all necessary presuppositions have been identified. Second, Dedekind does not formulate any fundamental principles for his logical system, i.e. laws upon which his constructions are based; much less does he provide a complete list. A third criticism, added in Volume 2 of *Grundgesetze*, is that Dedekind does not formulate any laws for 'abstraction' either, a fact that makes it unclear under what conditions the corresponding 'creation' of mathematical objects is allowed, including whether there are any limits for it. It is hard to deny, especially in retrospect, that Frege has put his finger on crucial points here. Yet his criticisms only establish that it is unclear whether a reduction of arithmetic to logic, as intended by Dedekind, has really been achieved, since his procedure contains gaps, while the possibility of closing those gaps is left open. (More on this point below.)

A basic constraint, or a systematic limit, in connection with Dedekind's or similar procedures is, presumably, consistency. Soon after formulating his criticisms of him, Frege was made painfully aware – in the form of Russell's antinomy – that his own law for introducing mathematical objects was inconsistent, namely Basic Law V. In Appendix II to *Grundgesetze*, Volume 2, he comments:

> *Solatium miseris, socios habuisse malorum.* [It is a comfort to the wretched to have company in misery.] I too have this solace, if solace it is; for everyone who in his proofs has made use of extensions of concepts, classes, sets[1], is in the same position. It is not just a matter of my particular method of laying the foundations, but of whether a logical foundation for arithmetic is possible at all. (*Frege 1997*, p. 280)

The corresponding footnote reads: '[1]Mr. R. Dedekind's "systems" also come under this head'. Here we have another close connection between Frege and Dedekind: the fact that their logical systems are equally susceptible to antinomies such as Russell's. Frege's overall reaction to Dedekind deserves closer attention, as do the ways in which his criticisms were appropriated and extended by later philosophers; but I have to reserve a further discussion of them for other occasions.[17]

[15] Schröder would soon endorse that pure mathematics 'is a branch of logic'; Hilbert was open to logicism initially, although he became critical later; Peirce was not a logicist himself, but thought of the line between logic and pure mathematics as evanescent. Cf. *Ferreirós 1999, 2009* for more.

[16] Cf. *Kreiser 2001*, pp. 295–296, also for other classes offered by Frege over the years.

[17] For more on criticisms of Dedekind by later philosophers, especially by followers of Frege and Russell, cf. *Reck 2013*. For Frege's own critical reaction to Dedekind, see *Reck forthcoming*.

4. Origins of logicism: Frege and nineteenth-century philosophy

Against the background provided so far, let us return to the issue of logicism's original motivation, thus also its significance. Overall, I want to juxtapose three general suggestions in this connection. The first we have already encountered (in Section 1): the view that logicism originated in Frege's works, and that it should be seen primarily as a response to Kant, Mill, etc., i.e. to inner-philosophical debates. A second proposal (discussed in the present section) is that, while Frege may have been the first to elaborate logicism systematically, the general approach goes back further in nineteenth-century philosophy, so that it has deeper roots in that sense. And as a third alternative (to be spelled out in Sections 5–7), there is the claim that logicism should be seen as rooted in certain developments in nineteenth-century mathematics. If understood in an exclusive way, these suggestions contradict each other. Instead, I will try to fit them into a more comprehensive whole.

The thesis that Frege's logicism, in particular, has roots in earlier philosophy arose in response to a largely a-historical treatment of Frege in analytic philosophy. More specifically, it was a reaction against Michael Dummett's influential interpretation, which included the claim that something totally new started with Frege. But as Hans Sluga, Gottfried Gabriel, and others argued, not only is such an a-historical view implausible in general, one can also identify specific influences on Frege. A central point of reference in this context is the nineteenth-century philosopher Hermann Lotze, one of whose classes Frege attended at the University of Göttingen. It is also known by now that Frege was familiar with Lotze's book, *Logik* (*1874*), in which we can read: '[Mathematics is] an independently progressive branch of universal logic'. Moreover, Bruno Bauch, Frege's colleague at the University of Jena, remarked in an article, entitled 'Lotzes Logik und ihre Bedeutung im Deutschen Idealismus' (*1918*), that Lotze's logic was of 'decisive importance' for Frege.

Lotze's and Bauch's remarks suggest an earlier origin of Frege's logicism in general. Beyond that, more specific logicist ideas have been located historically as well. For example, concerning the central and seemingly original Fregean view that a statement of number is a statement about a concept, Gabriel noted that it can already be found in the works of J.F. Herbart, another influential but often neglected philosophers from the nineteenth century. Then again, with respect to all such claims it seems fair to reply that, while Frege may indeed have found ideas and inspiration in these figures, it is only when put in the context of his logical system that the corresponding innovations acquire real force. For instance, while Lotze did suggest that we should see mathematics as 'part of logic', and while he even indicated that one has to go beyond Aristotelian logic in this connection, his specific understanding of logic lagged far behind Frege's, whose detailed, systematic treatments of formal inference, of classes, etc. remains unprecedented.[18]

While few signs of Lotze's and Herbart's influence occur in Frege's publications, one can find references to them in posthumously published manuscripts. However, there is a more prominent reference to yet another influential nineteenth-century philosopher in Frege's published works, namely, to F.A. Trendelenburg and his article, 'Über Leibnizens Entwurf einer allgemeinen Charakteristik' (originally published in 1857, republished in 1867). This points to an additional philosophical influence on Frege's logicism, namely, the rediscovery of Leibniz's work on logic in the nineteenth century. In this context, Frege's terminology is revealing too, such as his adoption of the Leibnizian terms *lingua characterica* and *calculus ratiocinator*. In fact, Frege characterizes his own 'conceptual notation' explicitly as a partial realization of Leibnizian ideas. There are also clear echoes of Leibniz's treatment of the natural numbers in Frege's approach to them, such as the

[18] For Herbart and Frege, cf. *Gabriel 2001*. For an evaluation of the debate between Dummett, Sluga, Gabriel, etc., including a judicious comparison of Lotze and Frege on logic, cf. *Heis 2013*.

recursive definition of the natural numbers based on the successor function ($2 = 1 + 1$, $3 = 2 + 1$, etc.).[19]

Another noteworthy aspect of Frege's references to Leibniz is that they start already in *Begriffsschrift* (*1879*) (cf. *Frege 1997*, p. 50, fn. B). In that text, he also already formulates his goal of showing that arithmetic can be based on logic alone, i.e. 'on the laws on which all knowledge rests' (*Frege 1997*, p. 48). Frege does not yet connect his project directly, or centrally, with Kant's analytic-synthetic distinction at this point in his development. (It is mentioned in §8 and §24, but in a way that suggests Frege's views about it to be still in flux.) Kant's distinction becomes prominent only in *Grundlagen* (*1884*). Why the shift? An explanation might be this: As Frege tells us later, he was quite disappointed with the initial reception of *Begriffsschrift*. He thus tried to motivate his logicist project further in *Grundlagen*, both by presenting it informally and by contrasting it with other philosophical views, such as Kant's and Mill's.[20] Soon after *Grundlagen*, the analytic-synthetic distinction, or at least the explicit use of these terms, becomes less prominent again.

5. Origins of logicism: Frege, Dedekind, and nineteenth-century mathematics

What this shift in terminology suggests is that Frege's goal was not from the start, and perhaps not primarily, to refute philosophers such as Kant and Mill. His explicit use of a Kantian framework in *Grundlagen* might be somewhat misleading in that respect. It could be part of a *post hoc* rationalization, prompted by the lack of positive reception of his work. This is not to say that Kant played no role for Frege at all, especially in his *Grundlagen* phase. Nor does it mean that Frege's logicism could, or did, not have deeper roots. What it makes room for, however, is to look for these roots elsewhere – in mathematics. After all, while Frege took one philosophy class with Lotze, his main training was in mathematics, as was his life-long professional appointment. A more specific alternative suggestion is, moreover, that Frege's – and Dedekind's – foundational works should be seen as in line with, or as the capstone of, the 'arithmetization of analysis', as pursued prominently by Cauchy, Bolzano, Weierstrass, and others in the nineteenth century.

The claim that the arithmetization of analysis is the main context, or at least a second crucial context, in which to see Frege's logicism is not new. In fact, Frege himself points in that direction too, as is well known (cf. *Grundlagen*, §§1–2). But this suggestion can be probed more deeply than is common, especially with respect to connections between mathematics and philosophy. If we do the latter, the question becomes what the motivation for the arithmetization movement in nineteenth-century mathematics was. A standard answer is that it was prompted by tensions, or even inconsistencies, within 'pre-arithmetized' forms of the Calculus, together with dissatisfaction about appeal to 'infinitesimals' in its early versions. Sometimes it is added that such issues threatened the certainty of mathematics. The remedy was, then, a more rigorous treatment of the real numbers and, eventually, the natural numbers, exactly as provided by Frege and Dedekind.[21]

Surely there is something right about this line of thought, although the part about certainty as the core motivation is questionable. (For one thing, it does not work well for the case of the natural numbers; and unlike Russell, Frege does not seem to be motivated much by skeptical worries, as already mentioned; nor does Dedekind.) But if certainty was not the

[19] For a rich account of the reception of Leibniz and his logic, cf. *Peckhaus 1997*.

[20] Frege's correspondence with A. Marty and C. Stumpf is illuminating here; cf. *Frege 1997*, pp. 79–83. Apparently it was Stumpf who suggested to Frege a more informal and philosophical approach.

[21] Cf. *Klein 1980* for a well-known account that emphasizes the issue of certainty in this context.

crucial issue, at least for Frege and Dedekind, what else was? Also, was there no connection to philosophical concerns then? A reply to the latter question that is prominent in the recent literature is to note a more general, and perhaps again Leibniz-influenced, rationalist tendency that informs Frege's approach, as evidenced, among others, by his continuing emphasis on 'Euclidean rigor' (cf. *Grundlagen*, §2, the Preface to *Grundgesetze*, Vol. I, etc.).[22] But once again, one may wonder whether this is part of a *post hoc* rationalization on his part. Also, does it tie Frege's general motivation enough, or in the right way, to mathematical developments? In any case, I now want to substantially enrich our story of logicism's mathematical origins, beyond quick appeals to arithmetization. And I want to build on our earlier discussion of Dedekind for that purpose.[23]

The phrase 'arithmetization of analysis' refers to the reduction of the fundamental notions of the Calculus, including those of limit, continuity, real number, and complex number, to arithmetic notions (in the narrow sense, i.e. involving only the natural numbers in the end). Like Frege, Dedekind puts his foundational writings explicitly in this context. For example, in the Preface to *Was sind und was sollen die Zahlen?* he mentions the goal of showing 'that every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers, – a declaration I have heard repeatedly from the lips of Dirichlet' (*Dedekind 1963*, p. 35). Besides G.L. Dirichlet, C.F. Gauss' treatment of the complex numbers is in the background here, i.e. his technique of treating them as pairs of real numbers.[24] Beyond justifying the use of complex numbers in general, it allowed 'higher analysis' to include a theory of complex-valued functions.

At the time, basing analysis on arithmetic was opposed to basing it on geometry, as we saw above. I already mentioned some methodological considerations that suggest such a reorientation, such as Dedekind's argument that we need arithmetic to get really clear about geometry, not *vice versa*. But there is also a direct link to the further goal of reducing arithmetic to logic – to 'the laws of thought', 'the laws without which to thinking is possible', or 'the laws on which all knowledge rests', as Frege and Dedekind put it. And as we will see, mathematical and philosophical concerns can be seen as intimately linked here (in a way that neither involves analyticity nor certainty, at least not directly). This becomes evident when we ask how mathematicians came to think about the difference between arithmetic and geometry during the period. Besides Frege and Dedekind, my three witnesses in this context will be: Gauss, Karl Weierstrass, and David Hilbert.

Let us begin with a passage from a letter written by Gauss, in the year 1830, in which he brings up a crucial epistemological difference as follows:

> [T]he theory of space has a completely different position with regards to our knowledge *a priori* than the pure theory of magnitudes. [. . .] We must humbly acknowledge that, whereas number is just a product of our minds, space also has a reality outside our minds, whose laws we cannot prescribe *a priori*. (Quoted in *Ferreirós 2007*, pp. 209–210)

Gauss' suggestion in this passage is that geometry, since it is about physical space, must have a different epistemological basis than 'the pure theory of magnitudes', where the latter includes his treatment of the complex numbers. Now, compare the following remark by Weierstrass, from lectures he gave in 1874:

[22] Cf. Part III of *Burge 2005* for a philosophically sophisticated elaboration of this suggestion. In *Weiner 2004*, the emphasis on Euclidean rigor is analyzed too, but in a less rationalist way.

[23] In what follows, I draw heavily on *Ferreirós 1999, 2007*. But I put his insights into a broader context, including by linking Dedekind's approach more closely to Frege's than he does.

[24] Gauss and Dirichlet both taught at Göttingen where Frege and Dedekind received their Ph.D.'s (and Dedekind did important editorial work for them); cf. *Ferreirós 1999* and *Tappenden 2006*.

> [W]e shall give a purely arithmetic definition of complex magnitudes. The geometric representation of the complex magnitudes is regarded by many mathematicians not as an explanation, but only as a sensory representation, while the arithmetical representation is a real explanation of the complex magnitudes. In analysis we need a purely arithmetic foundation, which was already given by Gauss. Although the geometric representation of the complex magnitudes constitutes an essential means for investigating them, we cannot employ it, for analysis must be kept clean of geometry. (Quoted in *Ferreirós 2007,* p. 211)

Weierstrass' concern seems more methodological, and less epistemological, than Gauss', although Gauss' earlier treatment of the complex numbers is brought up explicitly. Finally, we can find epistemological and methodological concerns combined in a related remark by Hilbert, from lectures he taught in 1891:

> Geometry [...] is essentially different from the purely mathematical domain of knowledge, like, e.g. number theory, algebra, function theory. The results of these domains can be obtained by pure thought, in that one reduces the facts asserted to simpler ones through clear logical inferences, until in the end one only needs the concept of whole number. [...] Today a proposition is only then regarded as proven, when in the last instance it expresses a relationship between whole numbers. (Quoted in *Ferreirós 2009*, p. 36)

There are several points in these passages that, while worthy of attention, must be put aside here (such as the reference to 'real explanations' by Weierstrass).[25] What is crucial, for my purposes, is the view that arithmetic and analysis, unlike geometry, belongs on the side of 'the purely mathematical domain of knowledge'.[26]

The widespread (but not uniform) adoption of that point of view in nineteenth-century mathematics, perhaps more than anything else, is what lies at the root of logicism – or so my main suggestion at this point. For Dedekind, in particular, logicism emerged as the attempt to think through systematically what should be seen as the nature and the basis of 'pure mathematics', as developed by his teachers Gauss, Dirichlet, etc. (and including number theory, analysis, algebra, 'function theory', etc., but not geometry). Moreover, while the need for such an attempt was tied intimately to methodological concerns for Dedekind – and therefore, connected with mathematical developments in a substantive manner – it was not unconnected with philosophical, especially epistemological, considerations, as Gauss' reference to '*a priory* knowledge', 'purity', etc. already indicates. Finally, Frege's logicism has deep roots in this vicinity too, as I want to make evident next.

Discussions of Frege's logicism usually start with his *Grundlagen*, sometimes also with *Begriffsschrift*. But here we have to go back further. Consider already the beginning of Frege's second dissertation (*Habilitation*), finished in 1874:

> When we consider complex numbers and their geometric representation, we leave the field of the original concept of quantity, as contained especially in the quantities of Euclidean geometry: its lines, surfaces, and volumes. (*Frege 1984*, p. 56)

[25] For more on explanation in mathematics, cf. *Mancosu 2008*, chapters 5–6, including further references. Compare also the remarks on explanation and understanding in *Reck 2009*.

[26] For more on the emergence of 'pure mathematics' in the nineteenth century, including Gauss' related philosophical interests, see *Ferreirós 1999* and, especially, *Ferreirós 2007*.

Frege then adds a striking remark about the development of mathematics:

> The concept [of quantity] has thus gradually freed itself from intuition and made itself independent. This is quite unobjectionable, especially since its earlier intuitive character was at bottom mere appearance. Bounded straight lines and planes enclosed by curves can certainly be intuited, but what is quantitative about them, what is common to lengths and surfaces, escapes our intuition. (*Frege 1984*)

And this leads him in the direction of views familiar from his later writings:

> A concept as comprehensive and abstract as the concept of quantity cannot be an intuition. There is accordingly a noteworthy difference between geometry and arithmetic in the way in which their fundamental principles are grounded. The elements of all geometric constructions are intuitions, and geometry refers to intuition as the source of its axioms. Since the object of arithmetic does not have an intuitive character, its fundamental principles cannot stem from intuition either. (*Frege 1984*, pp. 56–57)

What we have here – 5 years before the introduction of modern logic in Frege's *Begriffsschrift* (*1879*) and 10 years before the central use of Kantian terminology to motivate his project in *Grundlagen* (*1884*) – is an argument for logicism in the case of arithmetic. Its core is Frege's observation about the general, abstract nature of 'the concept of quantity'. For Frege too, the question becomes: What is its ultimate basis? Note again that Kant, Leibniz, and related philosophers are not yet mentioned here, while a Gaussian 'pure theory of magnitudes' looms large in the background.

6. Further aspects: Conceptual mathematics and Dedekind's logicist framework

Earlier I juxtaposed three suggestions for, or three general accounts of, the origins of logicism. Let me clarify the way in which I am proposing to look at them in this paper. My goal is not so much to show that they are wrong. Each is wrong if taken as an exclusive claim, I would say; but each also has something positive to contribute. Frege's project was indeed influenced by nineteenth-century philosophers such as Lotze, Herbart, and Trendelenburg; but with his systematic innovations in logic, he went far beyond them. Frege did present his work as relevant to philosophical debates going back to Mill, Kant, and Leibniz; yet it had deeper, more mathematical roots too. Frege's logicist project can be seen as a continuation of the arithmetization of analysis; but as I argued, it is only by probing the motivation for the latter more deeply, and especially, by considering the rise of 'pure mathematics', that its full significance reveals itself. Finally, Dedekind's case is interesting because it helps to bring those connections to the fore; and conversely, by taking them seriously Dedekind can be rediscovered as an important figure in the rise of logicism.

For further clarification, let me reformulate the outcome of my discussion so far slightly. Given all the historical evidence we have by now, any one-dimensional or mono-causal story about the origins of logicism should be seen as implausible.[27] Moreover, if we recognize the various factors involved, it becomes evident that any strict division between 'inner-philosophical' motivations for logicism, on the one hand, and 'inner-mathematical' motivations, on the other hand, is untenable. What the rise of 'pure mathematics' brought with it was not only methodological but also epistemological concerns. The latter did not really, or not for everyone, have to do with certainty; nor did they primarily, or immediately, involve the notion of analyticity.[28] It was more the *a priori*/*empirical* distinction

[27] Thanks to Jeremy Heis for helping me formulate the results of my discussion in this way.

[28] The qualifier 'not immediately' is important, since explorations of the *a priori*/empirical distinction often bring back the notion of analyticity, as Clinton Tolley has reminded me correctly.

that played a role; and it involved the separation of 'pure mathematics' from geometry, for methodological and other reasons. It is in this sense that philosophical reflection and mathematical practice can be seen as more intimately connected, and the significance of logicism as richer, than is acknowledged in the three standard accounts.[29]

But actually, the theme of close ties between philosophy and mathematics in this context can be pursued a few steps further, especially in connection with Dedekind. Doing so will lead us beyond the aspects of the emergence of 'pure mathematics' considered so far, including the issue of *apriority*, although it will not be wholly unrelated. It will also lead us back to the question of what should be included under 'logic' from a logicist perspective (both historically and philosophically speaking). In the present section, I want to bring to bear the rise of a more 'conceptual' methodology in nineteenth-century mathematics, also beyond foundational works, than what was usual before. Gauss will again be a central figure here, but also Dirichlet and Bernhard Riemann.[30] In the next section, I will turn to the nascent use of classes in nineteenth-century mathematics, together with the introduction of a variety of novel mathematical entities during that period.

In my summary of Dedekind's foundational writings above, I mentioned that it was crucial for him to distill out basic concepts so as to ground arithmetic and analysis in them, rather than to rely on 'geometric evidences'. We also considered some methodological, and related epistemological, reasons for doing so. In his more mainstream mathematical writings, another source for Dedekind's 'concept-based' approach emerges. Thus, in his work on algebraic number theory he quotes the following remark by Gauss approvingly: 'In our opinion [. . .] such truths should be extracted from concepts rather than from notations' (*Dedekind 1930–1932*, p. 54, my translation). In the same context Dedekind refers to Riemann's lectures on complex analysis as well; in them mathematicians were urged to base their theories on 'characteristic inner properties' of the objects under study, not on 'external notations' (p. 55). Finally, Dirichlet, after whose lectures on number theory Dedekind often discussed methodological issues with him, reinforced the same idea. In all these cases, the operative contrast is not with a mathematical approach based on geometry, but rather, with a notation-based, formalist approach to it (like Weierstrass', Riemann's main foil). Where Dedekind went further than his teachers was in approaching the foundations of elementary arithmetic in the same way.[31]

One can even argue that Dedekind's logicism consists precisely in the attempt to derive arithmetic and analysis from core concepts, as opposed to relying either on geometric evidences or on empty formalisms.[32] In the case of analysis, he makes the concepts of field, ordering, and continuity (line-completeness) basic; for arithmetic, those of infinity and simple infinity play the same role. This amounts to a form of logicism insofar as Dedekind's key concepts are defined solely in terms of 'logical' notions – those of object, set, and function. Crucially, he works with generalized notions of set and function here, where any 'arbitrary' function on the natural numbers and any 'arbitrary' subset of them counts, not just those definable formalistically or representable intuitively. Dedekind defends the

[29] Note here, also, that Herbart or Lotze may well have been aware of relevant developments in mathematics; after all, not only Kant's shadow loomed large in Germany at the time but also Gauss', especially in Göttingen, where both taught for a while. (This is speculation, i.e. I do not have any concrete evidence.) For Gauss' critical relation to Kant's philosophy, see again *Ferreirós 2007*.

[30] For earlier discussions of this theme, cf. *Stein 1988* and, before that, *Cassirer 1910*; more recently, see *Tappenden 2005, 2006, 2008*, *Reck 2008/2011, 2009*, partly also *Klev 2011*.

[31] For the role of Gauss and Dirichlet in this context, cf. *Stein 1988* and *Reck 2008/2011, 2009*; for more on Riemann, as connected to both Dedekind and Frege, cf. *Tappenden 2005, 2006, 2008*.

[32] For a detailed, philosophically subtle articulation of this suggestion, cf. *Klev 2011*.

resulting methodological 'freedom', including the use of non-computable functions and actually infinite sets, also against constructivist criticisms, as voiced by Leopold Kronecker at that time. The core of Dedekind's defense is to point to the fruitfulness of the new approach (cf. *Dedekind 1963*, p. 45, fn.*).[33]

Now, the specific manner in which Dedekind derives results from his basic concepts is by reasoning about objects, or whole systems of objects, that fall under them. For example, he proves theorems in arithmetic by studying the general properties of systems that fall under the concept of simple infinity (e.g. that they allow for proofs by mathematical induction). Similarly, he proves results in analysis by considering systems that fall under the concept of complete ordered field (e.g. the mean value theorem). This presupposes, of course, that there are some basic objects to reason with; it also presupposes the existence of various relations and functions on them; and it presupposes that certain subsequent constructions are feasible (e.g. that of the system of cuts). It is at this point where questions about the nature and scope of the logic used by Dedekind become urgent. Clearly 'logic' means not just a general system for defining concepts for him, but one in which crucial existence assumptions and corresponding constructions are grounded as well.

If we compare Dedekind and Frege further in this context, there are important differences, no doubt; but there are also striking commonalities. To begin with, Frege too was influenced by the 'conceptualism' of Riemann and others, i.e. his logicism has roots there as well.[34] Among others, note that a main goal for him was to devise a 'concept script' in which reasoning, or 'calculating', about concepts can be done rigorously. Note, also, that for both Frege and Dedekind 'logic' comes to include a theory of, or at least a general framework for, sets (or 'classes') and functions ('mappings', including Fregean 'concepts'). Frege's approach can more properly be called a 'theory', since he states his 'basic laws' for them explicitly, while Dedekind leaves them implicit. Still, the two do not only agree on the general scope of logic, but also on specific details, e.g. on treating sets and functions extensionally. This leads to the question of where their shared conception of logic comes from, especially since it does not square with the traditional view, held by Kant and others, on which logic does not deal with any specific objects, or even, with objects at all. Nor does it square with the twentieth-century view that logical truths are those true in any domain. Another way to ask the same question is: Why were both Frege and Dedekind led to the idea of basing 'pure mathematics' on logic conceived of in this particular, inclusive way?[35]

7. Further aspects: New mathematical objects and the systematic use of classes

For answering the questions just raised, pointing towards philosophers such as Lotze, Herbart, or Trendelenburg does not help much, at least in itself; nor does a quick appeal to arithmetization. Instead, I now want to invoke some further developments in nineteenth-century mathematics.[36] This will include: the rise of projective geometry, and more particularly, the introduction of 'ideal elements' within it; parallel innovations in algebraic number theory, such as the appeal to 'ideal divisors'; and several related developments

[33] For more on the notion of 'arbitrary' sets and functions, including further ties to Dirichlet's work, cf. *Ferreirós 2011*; for mathematical 'freedom', including connections to Georg Cantor's work, cf. *Tait 1997*, *Reck 2003*; and for the issue of 'fruitfulness', cf. *Tappenden 2006, 2008*.

[34] In *Tappenden 2005, 2006, 2008*, this background of Frege's approach is documented in detail.

[35] I am grateful to Danielle Macbeth for pushing me on this issue. I also owe to her the point that Frege's introduction of his 'concept script' is directly related to his Riemannian 'conceptualism'.

[36] In doing so, I will draw on *Ferreirós 1999*, *Tappenden 2005, 2006, 2008*, and *Wilson 1992, 2010*. But closely related ideas can already be found in *Cassirer 1910*; cf. *Heis 2011*.

in geometry, algebra, analysis, and even in 'applied mathematics'. In connection with them too, I will illustrate how philosophical reflections can grow out of mathematical practice.[37]

At the time when Frege and Dedekind were trained as mathematicians, projective geometry was already established as a new part of mathematics (by J.-V. Poncelet, Karl von Staudt, etc.). However, the postulation of certain 'ideal elements' within it – novel objects that did not fit into a traditional conception of geometry – had caused controversy. This includes both 'points at infinity', where parallel lines are supposed to meet, and 'points' characterized by complex coordinates, which also go beyond the intuitably accessible. Working with these new objects was very fruitful mathematically; it allowed for uniform, generalized, and appealingly systematic treatments of otherwise disconnected results. But how should their nature be understood, as well as their introduction justified? A purely formalist answer, where terms seemingly referring to such objects are treated as empty but computationally useful signs, was one option. This was similar to how 'imaginary' numbers had been dealt with previously, although that had just changed with Gauss' re-conceptualization of complex numbers. A suggestion more in line with Gauss' work was to treat them as 'objects of pure thought', or more specifically, as derivative of concepts in some sense – as a kind of 'concept-objects'.

There are three connections to Frege here. First, some of his pre-*Begriffsschrift* writings in mathematics were concerned with representations of the complex numbers, as growing out of Gauss' work; he was thus clearly familiar with the basic debates. Second, his discussion of the notion of 'direction' for lines in *Grundlagen* (§64) corresponds exactly to the introduction of points at infinity in projective geometry, thus suggesting a connection between projective geometry and his work on the foundations of arithmetic. Third, there is the intriguing but controversial proposal that the Frege of *Grundlagen* had not yet fully settled on how to treat the natural numbers – his 'concept objects' – while suggesting the use of 'extensions of concepts' only tentatively; this changed, of course, with *Grundgesetze*.[38] But before exploring these issues further, let me add a brief summary of parallel developments in algebraic number theory, with direct ties to Dedekind's work in turn.

While nineteenth-century geometers employed objects such as 'points at infinity' to generalize and systematize results in projective geometry, E.E. Kummer introduced 'ideal divisors' into algebraic number theory to build on earlier work, again by Gauss and others, on the solvability of Diophantine equations. Here too, the move proved not only fruitful but also controversial – Kummer's novel 'ideal numbers' seemed as mysterious as the 'ghostly' points employed in projective geometry. Besides questions about their nature and existence, there was the more practical issue of how far his (piecemeal) technique could be extended, i.e. what the limits of, and potential problems with, these peculiar domain extensions were. In Dedekind's celebrated contributions to algebraic number theory, he addressed both questions successfully. He did so by replacing Kummer's 'ideal numbers' by his 'ideals'. The latter are infinite sets of familiar numbers, and thus, parallel to the cuts used in Dedekind's reconceptualization of the real numbers. Crucially, the employment of infinite sets led to a unified, systematic theory in both cases. Moreover, Dedekind used related techniques (quotient constructions etc.) elsewhere too, e.g. in his innovative treatment of modular arithmetic and of Galois theory in algebra.

[37] I do not claim that what follows provides a comprehensive answer to my question above, one in which all aspects are explored fully; but it should at least provide a substantive partial answer.

[38] Cf. *Wilson 1992, 2010* for more. Wilson argues (controversially) that Frege may have changed his mind on this issue even while writing *Grundlagen*, in a way that left traces in the text.

The reference to Dedekind's work in algebra leads to a whole nest of further nineteenth-century developments (all loosely related) that are also relevant here. This includes: the joint treatment of propositional logic and classes in Boolean algebra; the introduction of further 'number systems', from Hamilton's quaternions through Grassmann's vector algebras to Cantor's transfinite numbers; the study of novel geometric spaces, especially Riemann's theory of 'manifolds'; and even, the investigation of more general 'spaces', such as the 'space of colors', by Riemann, Helmholtz, and others.[39] While I cannot go into detail with respect to any of these developments, three general points are fairly clear. Namely, in those cases too mathematicians considered novel objects; they were again willing to go far beyond the geometrically intuitable; and the consideration of whole systems of objects, infinite classes, and generalized functions played a crucial role as well. Finally, in George Boole's work, classes were explicitly seen as part of 'logic', a view that influenced Peano, Schröder, and Russell, among others, directly.[40]

Let us return to Frege once more in this context. Throughout his writings one can find sharp criticisms of earlier views about 'agglomerations', 'totalities', or 'sets' (often understood, somewhat inchoately, in a mereological sense). And he is right that a better, more systematic theory of classes was needed at the time.[41] This may give one the impression that he was breaking radically with earlier views. Frege's theory of 'extensions of concepts' is certainly original, in many ways even unique. At the same time, he also appeals to a shared understanding – shared with the Booleans, among others – that a theory of classes, if understood properly, should be seen as part of 'logic'. What about the suggestion that Frege seems somewhat tentative about the use of classes up to *Grundlagen*, while from *Grundgesetze* on, they have become central for him? Well, consider this: Isn't it striking that it was precisely after a close study of Dedekind's essay, *Was sind und was sollen die Zahlen?*, that Frege started to treat classes as crucial or even indispensable, including spelling out a systematic theory for them in *Grundgesetze*? If so, perhaps Dedekind's work played an important role in convincing him that this was the way to go.[42]

In any case, the use of classes was 'in the air' at the time. Dedekind was one of the first, arguably even the first, to use infinite classes systematically in mathematics. He did so not only in his essays on the natural and real numbers, but also in his ideal theory and in earlier algebraic works.[43] He thus contributed, among others, to the clarification of the development in nineteenth-century mathematics mentioned above. In fact, his use of cuts in connection with the real numbers and of ideals in algebraic number theory can be seen as paradigms for how to re-conceptualize novel mathematical objects. Frege's treatment of the natural numbers in *Grundlagen* and his sketched account of the real numbers in *Grundgesetze* fit in as well. (Both are equivalence-class constructions, using infinite classes essentially.) Along such lines, both Frege's and Dedekind's logicism reveals itself as a response to, and an

[39] For more on Riemann, Hamilton, Grassmann, Cantor, and Boole, cf. *Ferreirós 1999*. The claim that even developments in 'applied mathematics', such as the study of color space, contributed to the rise of 'pure mathematics', is defended in Mark Wilson's work; cf. also *Maddy 2008*.

[40] Within British logic, this view goes back to at least Whatley; cf. *Heis 2012*. For more on related developments in Germany, including Leibniz' role in them, cf. again *Peckhaus 1997*.

[41] For an older discussion of Frege's criticisms and of related developments in his own position, cf. *Burge 1984*. However, Burge does not consider the mathematical context enough yet, I think.

[42] Cf. *Sundholm 2001* for this suggestion. In *Wilson 1992, 2010*, a close connection between Dedekind and Frege in this context is suggested as well. In *Ferreirós 1999, 2011, forthcoming*, on the other hand, the differences between their approaches to classes are emphasized.

[43] As Akihiro Kanamori writes about Dedekind's work on modular arithmetic (from the 1850s): 'One can arguably date the entry of the actual infinite into mathematics here, in the sense of infinite totalities serving as unitary objects within an infinitary mathematical system' (*Kanamori 2010*).

extension of, a variety of developments in nineteenth-century mathematics, including but not only the arithmetization of analysis.[44]

What both Frege and Dedekind provided, then, was a systematic reflection on the tools employed in 'logically' re-conceiving (old and new) mathematical objects. In doing so, each of them went from using various particular classes and functions to a general, self-contained study of them.[45] Now again, was there one main motivation for that move? Was it, for instance, to defend traditional parts of mathematics, such as analysis, against worries about their consistency? Or was it to justify new additions, like Dedekind's theory of ideals, against basic doubts, as has also been suggested? Such worries and doubts may have played some role; but they are not emphasized much (at least before the discovery of the set- or class-theoretic antinomies). More prominent, in Dedekind's as well as Frege's writings, are other methodological concerns (fruitfulness, generalizations and their limits, the explicit articulation of presuppositions, the articulation of core concepts, etc.). Those have roots in earlier mathematics (by Gauss, Riemann, Dirichlet, etc.), but they also exhibit a more broadly epistemological character. In any case, I would again resist any one-sided, mono-causal account – not just one road led toward the new 'logic'.[46]

8. Frege, Dedekind, and the varieties of (neo-)logicism

Even if one is open to the claim that Frege's and Dedekind's logicist projects have deep roots, not only in nineteenth-century philosophy, but also, and especially, in nineteenth-century mathematics, one may wonder whether it is justified to talk about a general theory of classes and functions as part of 'logic'. After all, are not both Frege and Dedekind relying on strong existence assumptions; and is not logic supposed to be free of reference to any particular objects, or free of existential commitments? As a quick reply, one can point out that this challenge is based on assumptions about 'logic' that were not universally shared at the time and that can be questioned. Frege and Dedekind, but also Boole, Peano, Schröder, etc., accepted that 'logic' includes a theory of classes. For them the basic question – the question of the feasibility of 'logicism' – was not whether mathematics can be developed without any existence assumptions, but whether a reduction of it to a general theory of classes and functions is possible. But more needs to be said in this context, not just from a historical but also from a systematic point of view.

At this point, a lot depends on how the notion of class – or 'extension', 'system', 'set', etc. – is understood. (Similarly for the notion of function; more on it later.) It also matters what precisely we mean by 'mathematics', in addition to 'logic'. To begin with, one can see logic not only as the formal study of deductive systems, as is often done today, but also as providing a theory of concepts, judgment, and inference, understood in a more inclusive and substantive sense. Indeed, the latter was the traditional view. It was also the view adopted by Frege, especially in his later writings, i.e. those in which he adopts a conception of classes as 'extensions of concepts'. More specifically, for Frege classes are derivative of concepts – they are true 'concept-objects' – which is what makes them 'logical'. (There is a further question about whether such a 'logical notion of class' is

[44] This is not to deny that there are important differences between their two versions of logicism. One difference is that Frege makes the cardinal notion of number basic; he also identifies the natural and real numbers with infinite classes; in contrast, Dedekind treats both structurally; and he makes the ordinal aspect primary for the natural numbers; cf. *Reck 2005, 2003*, respectively.

[45] In the terminology of *Wilson 2010*, Frege's and Dedekind's systematic reflections on the use of classes and functions constitute the move from 'relative' to 'absolute logicism'.

[46] Cf. the title of *Wilson 1992*. I thus disagree with Wilson's implicit suggestion of one 'royal road' to logicism; but I take his observations to be an important part of the overall story.

adequate to capture all of modern mathematics, since the latter requires ‘arbitrary’ classes or sets.)[47]

Let me probe Frege’s views about classes more deeply, since they are subtler than just the basic claim that they are derivative of concepts. For one thing, Frege (unlike, e.g. Russell) adopts an extensional view of concepts, on the basis of his sense-reference distinction.[48] For another thing, Frege studies various equivalence relations on concepts, as well as corresponding ‘concept-objects’. Classes turn out to be the objects we get for the most fine-grained equivalence relation (for concepts conceived of extensionally), while numbers result for a more coarse-grained relation (equinumerosity). In the background is, pretty clearly, the technique of introducing new entities via equivalence relations, even if mathematicians today usually do not apply it to concepts. Note, furthermore, that Frege’s ‘basic law’ for classes concerns exactly the most fine-grained case. What he suggests, then, is to reduce all other relevant cases to it. He thus attempts to set up a framework for all foundationally important constructions in one swoop, as opposed to having to introduce new mathematical objects piecemeal along the way.[49]

I take Dedekind’s view of classes (‘systems’) to be related to but not identical with Frege’s. For Dedekind, the mathematical notion of function is central, in a way that differs subtly from Frege’s position. Frege too starts with functions, as concepts end up being a generalized kind of functions (truth-valued ones). But for him all concepts, indeed all functions, have to be defined for all objects, at least officially.[50] Dedekind, in contrast, works with functions that have restricted domains and ranges, as is usual in mathematics. Indeed, I take Dedekindian classes simply to be the domains and ranges of functions, so that there is a sense in which they are derivative for him too. The most basic notion is that of functionally relating objects of various kinds; as he puts it (in a passage already quoted partly earlier):

> If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensible foundation [. . .] must, in my judgment, the whole science of numbers be established. (*Dedekind 1963*, p. 32)

The way in which Dedekind talks about the fundamentality of the notion of function in this passage may sound too subjective from a Fregean point of view (since it is put in terms of ‘abilities of the mind’). However, one does not have to read him as adopting a problematic psychologistic notion of function. Arguably, functions are as ‘objective’ for him as for Frege; or at least, they are as ‘objective’ as mathematics needs them to be. In any case, I take Dedekind to conceive of ‘logic’ as including the systematic study of functions (and their domains, ranges, etc.). What is its relation to ‘mathematics’, then? Well, in the

[47] See again *Ferreirós 2011* on the issue of ‘arbitrary’ sets or classes. However, Ferreirós is not charitable enough to Frege, I think, partly because he reads him too much through Russellian eyes.

[48] Might Frege have been able to accept arbitrary classes, via arbitrary functions, even if he did not see the significance of such a move himself yet? I am not sure; but I think this possibility is worth considering. For a very different reading, see *Hintikka and Sandu 1992*; but cf. *Burgess 1995*.

[49] From this perspective, it is interesting that Fregean neo-logicists retreat to proceeding piecemeal with their abstraction principles, since Frege’s original route seems blocked by Russell’s antinomy.

[50] But see the discussion of Frege’s alleged ‘sharp boundaries’ requirement in *Blanchette 2012*, especially Ch. 3, where his procedure is brought more in line with ordinary mathematics.

latter, we study particular functions, or specific kinds of functions, while logic provides a general framework for doing so.[51]

Dedekind's remark that functionally relating things to each other constitutes an 'absolutely indispensable foundation', and that without it 'no thinking is possible', allows for another interesting comparison to Frege. At certain points, both of them view 'logic' as the discipline in which notions and techniques shared by, and indispensable for, thinking in general are investigated; or perhaps better, for all scientific thinking (rigorous, systematic, deductive, etc.).[52] Insofar as the notion of function, and with it the notion of class, has that status, it provides another way of seeing that it falls within 'logic'. It also motivates further why Dedekind and Frege adopt such broad notions of logic, including theories of classes. Post-Fregean logicians may balk at such breadth. But the restriction of logic to, say, first-order logic is not uncontroversial either. Also, not too much should hang on the word 'logic', in itself, in this context. More important would seem to be what the basic principles of the discipline at issue are and how they can be justified.

This brings us back to Frege's main criticism of Dedekind: that he does not spell out the fundamental laws for his logical system explicitly, much less investigate their epistemological source further. Dedekind's approach would have to include laws governing all the existence assumptions in his constructions, concerning both classes and functions. (As he does not reduce functions to classes, separate laws will be needed for them.) In addition, there would have to be laws for the kind of 'abstraction' used by him, but not by Frege, when he introduces the natural and real numbers along structuralist lines. But actually, neither Dedekind nor Frege work with explicitly formulated existence assumptions at bottom. For Frege, the crucial law for classes is his infamous Basic Law V, which does not have the form of an existential statement. Or course, it implies such statements in the context of his logical system, as he is well aware.[53] In Dedekind's case, all corresponding laws remain implicit, although they can be partly gleaned from his constructions.

Another general observation in this context is that, while neither Dedekind nor Frege proceed thoroughly enough from a later perspective, it was only via their and similar works that it became clear just how careful one has to be, and in particular, that paradoxes lurk in a totally unrestricted notion of class. Both Frege and Dedekind were shocked when they became aware of the corresponding paradoxes (Russell's, also Burali-Forti's, etc.). The most successful and widely accepted response to them, in later work, was Zermelo-Fraenkel set theory. But with respect to that theory, the view is usually that it is no longer a part of 'logic'. Then again, it is not obvious on what that view can be based. Is the argument that the ZF axioms are in some sense too 'constructive'; but why can logic not be that way? Or is it that they are not 'simple', or not few enough in number, in contrast to one 'logical' comprehension principle? Again, that seems not entirely clear. Perhaps ZF set theory relies on 'intuitive' consideration in the end; but if so, how? Presumably even the latter would not involve traditional geometric intuition, would it?[54]

51 Insofar as category theory provides a general theory of functions, it can be seen as playing a similar role for mathematics. Indeed, Dedekind's emphasis on the notion of function, his 'conceptual' approach, and related aspects put him in close proximity to category theory.

52 This is not meant as a defining characteristic of logic, since it is too vague. (Whether 'generality' plays such a role for Frege has led to much discussion in the secondary literature on him.) Still, I believe that the quasi-transcendental considerations at play here deserve to be explored further.

53 Then again, the crucial implications in Frege's case involve what from today's point of view would be instantiation rules; and he does not formulate those as explicit logical laws either.

54 For an earlier discussion that raises similar questions concerning set theory, cf. *Parsons 1967*. I take Frege's remarks about the epistemological source for his basic logical laws, put partly in terms of his notion of sense, to be relevant. But a further discussion of that aspect is not possible here.

Let me add one more consideration in this context, so as to connect Dedekind to Frege in yet another way. In recent neo-logicism, the proposal for how to deal with the basic existence claims needed for mathematics is to rely on Fregean 'abstraction principles'. Thus, Hume's Principle guarantees the existence of a countably infinite class, while others provide for classes of higher cardinalities. An additional and more philosophical claim is, then, that these abstraction principles are 'quasi-definitional'. This is presented as an argument for considering them as 'logical'. But the issue is contested, partly because the notion of 'definition' at play remains controversial. Now, from a Dedekindian point of view a different, more congenial response is possible too: Why not adopt such Fregean principles for Dedekind's purposes, and in particular, so as to take the place of his infamous 'proof' of the existence of an infinite set (Proposition 66)? If so, it might allow for a partial neo-logicist rehabilitation of Dedekind's approach too, not just of Frege's.[55]

To recover a fully Dedekindian form of neo-logicism, it would not be enough to adopt Fregean abstraction principles, however. Recall that Frege also complained, rightly, about the lack of principles for 'Dedekind abstraction'. But again, the observation that Dedekind did not provide the latter does not establish, in itself, that they cannot be supplied for him retroactively. Antinomies such as Russell's do not obviously rule that out either (especially since Dedekind abstraction is not 'inflationary', in the sense of raising the cardinality of the domain of objects). This suggests a general program for developing a Dedekindian form of neo-logicism – on a par with neo-Fregean forms – namely: spell out all the needed basic principles, together with motivations for them (their epistemic sources, connections to mathematical practice, etc.). As one benefit, this might lead to new mathematical results. As another benefit, the remaining differences to a neo-Fregean approach (e.g. the structuralist aspect) might then be interesting to explore further philosophically.[56]

Yet another reason for exploring a Dedekindian, or neo-Dedekindian, form of neo-logicism is the following: Frege's and Dedekind's original forms of logicism were both rooted deeply in mathematical practice. In fact, they were rooted in it in similar ways, as I argued, even if the positions were then developed in different ways.[57] However, awareness of these mathematical roots has been lost in many recent discussions of logicism, while ties to inner-philosophical debates have been overemphasized. Pushing things in a more Dedekindian direction (while also using resources from neo-logicism, set theory, category theory, etc.) could help to restore the balance. It would also illustrate further that studying the foundations of mathematics, on the one hand, and studying mathematical practice, on the other hand, need not be as separate as is sometimes assumed today.

9. Summary and concluding remarks

It was not the goal of the present paper to fully elaborate a Dedekindian, or neo-Dedekindian, form of logicism, although parts of my discussion were meant to motivate such a project. My primary goal was, instead, to clarify the origins of logicism, together with rehabilitating Dedekind as one of its main representatives. While logicism is still

[55] In *Demopoulos and Clark 2007*, the fact that a Fregean, or neo-Fregean, approach leads to a proof of relevant existence claims is seen as a major advantage over Dedekind. What I am suggesting here, very briefly, is to see the two sides as complementary; cf. Reck (forthcoming) for more.

[56] As a further similarity to neo-logicism, note that Dedekind calls the part where he introduces his version of the natural numbers a 'definition' ('Definition 73' in his 1888 essay). Might 'Dedekind abstraction' thus be seen as 'quasi-definitional' too? That seems worth exploring further too.

[57] Once again, I am not denying that important differences between their two positions remain. Another difference concerns 'Frege's Constraint'; cf. *Wright 2001* and the related discussion in *Reck 2005, 2013*. But again, such differences should not make us ignore the commonalities.

often thought of as a response to debates within philosophy, I have emphasized its mathematical roots. This was not meant to show that it only has mathematical significance, nor that philosophy played no role in its rise at all. Rather, I wanted to display the depth and multi-dimensionality of its background. I also wanted to make evident that, and how, philosophical concerns grew out of mathematical practice in this case. Significantly, logicism revealed itself as in line with, or as the completion of, the rise of 'pure mathematics', i.e. the separation of arithmetic, analysis, algebra, etc. from a geometric foundation.

Neo-Fregean philosophers have been remarkably successful in reviving interest in logicism within contemporary analytic philosophy. From the point of view of this paper, a weakness of that revival is, however, that it has led to a strong, and often exclusive, focus on inner-philosophical issues as far as the significance of logicism is concerned, revolving around 'analyticity' or closely related notions. Yet there are various questions, doubts, and implicit disagreements about those notions (in their Kantian, Fregean, Russellian, Carnapian, neo-Fregean, and other forms). Also, while Frege used the 'analytic/synthetic' terminology in some of his writings, this is not a constant. In both earlier and later works, he characterized logicism more generally, as the reduction of arithmetic to 'laws of thought'. In the present paper, analyticity was not taken as central and basic. Instead, both Frege's and Dedekind's talk of 'laws of thought' was explored with respect to its mathematical roots.

Besides their elusiveness, there is another disadvantage of tying logicism too intimately, or too exclusively, to inner-philosophical discussions. Namely, often those discussions end up having little to do with either the history of mathematics or current mathematical practice. By focusing on Dedekind's case, in contrast, we can see how intimately logicism was tied to mathematical developments originally. The more we recognize it as connected with the rise of 'pure mathematics', with the emergence of the 'conceptual' methodology typical for much late nineteenth- and twentieth-century mathematics, etc., the more the full significance of logicism can be recovered. As an additional benefit, Fregean neo-logicism can now be judged by how much, or how little, it contributes to such issues. And an alternative Dedekindian version might score better in this connection, depending on its details.

In the end, a question remains about the basic principles to be used in grounding 'pure mathematics' – be they neo-Fregean abstraction principles, parallel Dedekindian principles, the axioms of standard set theory, or, say, corresponding principles of category theory. What projects such as Dedekind's and Frege's provided were many insights into what kind of principles could, and could not, be used for that purpose. This constitutes a very positive, lasting legacy, also beyond 'logicism'. But how should we think about their ultimate justification, whether we call them 'logical' or not?[58] I have not fully answered that question here; nor did Frege and Dedekind in their original works. But I hope to have shown how closely it was connected with mathematical practice originally, not just with inner-philosophical debates. Not loosing that connection would seem to be a good thing for the philosophy of mathematics, in this context and more generally.[59]

[58] Put slightly differently, the question here is the following: How much, or in what sense exactly, are the principles at issue meant to provide a 'foundation' for mathematics. There is a direct line from Dedekind to Hilbert in that respect; for the early Hilbert, cf. *Ferreirós 2009*. Even later, more pragmatically oriented approaches to mathematics, such as Carnap's or those of current 'naturalists', face this question in the end (they just provide more deflationary answers). I plan to return to this general issue – in connection with Frege, Dedekind, and Cassirer – in future work.

[59] Earlier versions of this paper were presented at the APA/ASL meeting, San Diego, CA, April 2011, at the conference *Frege's Philosophy of Mathematics and Language*, Center for Logic and Philosophy of Science, Bucharest, Rumania, May 2011, and at a meeting of the *History and Philosophy of Logic and Mathematics* group, Riverside, CA, January 2012. I am grateful to audience members at all three events for criticisms, encouragements, and other comments, especially Sorin Costreie, Jeremy Heis, Danielle Macbeth, and Clinton Tolley. Thanks also to an anonymous referee for this journal. Research on the paper was supported by NSF Scholar's Grant SES-0914461.

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